CHAPTER 1

Theory of Oscillators

1.1 Introduction

There are many types of oscillators, and many different circuit configurations that produce oscillations. Some oscillators produce sinusoidal signals, others produce nonsinusoidal signals. Nonsinusoidal oscillators, such as pulse and ramp (or saw-tooth) oscillators, find use in timing and control applications. Pulse oscillators are commonly found in digital-systems clocks, and ramp oscillators are found in the horizontal sweep circuit of oscilloscopes and television sets. Sinusoidal oscillators are used in many applications, for example, in consumer electronic equipment (such as radios, TVs, and VCRs), in test equipment (such as network analyzers and signal generators), and in wireless systems.

In this chapter the feedback approach to oscillator design is discussed. The oscillator examples selected in this chapter, as well as the mix of theory and design information presented, help to clearly illustrate the feedback approach.

The basic components in a feedback oscillator are the amplifier, an amplitude-limiting component, a frequency-determining network, and a (positive) feedback network. Usually the amplifier also acts as the amplitude-limiting component, and the frequency-determining network usually performs the feedback function. The feedback circuit is required to return some of the output signal back to the input. Positive feedback occurs when the feedback signal is in phase with the input signal and, under the proper conditions, oscillation is possible.

One also finds in the literature the term negative-resistance oscillators. A negative-resistance oscillator design refers to a specific design approach that is different from the one normally used in feedback oscillators. Since feedback oscillators present an impedance that has a negative resistance at some point in the circuit, such oscillators can also be designed using a negative-resistance approach. For a good understanding of the negative resistance method, a certain familiarity with oscillators is needed. That is why the negative resistance method is discussed in Chapter 5.

1.2 Oscillation Conditions

A basic feedback oscillator is shown in Figure 1.1. The amplifier’s voltage gain is $A_v(j\omega)$, and the voltage feedback network is described by the transfer function $\beta(j\omega)$. The amplifier gain $A_v(j\omega)$ is also called the open-loop gain since it is the
gain between $v_o$ and $v_i$ when $v_f = 0$ (i.e., when the path through $\beta(j\omega)$ is properly disconnected).

The amplifier gain is, in general, a complex quantity. However, in many oscillators, at the frequency of oscillation, the amplifier is operating in its midband region where $A_v(j\omega)$ is a real constant. When $A_v(j\omega)$ is constant, it is denoted by $A_{vo}$.

Negative feedback occurs when the feedback signal subtracts from the input signal. On the other hand, if $v_f$ adds to $v_i$, the feedback is positive. The summing network in Figure 1.1 shows the feedback signal added to $v_i$ to suggest that the feedback is positive. Of course, the phase of $v_f$ determines if $v_f$ adds or subtracts to $v_i$. The phase of $v_f$ is determined by the closed-loop circuit in Figure 1.1. If $A_v(j\omega) = A_{vo}$ and $A_{vo}$ is a positive number, the phase shift through the amplifier is $0^\circ$, and for positive feedback the phase through $\beta(j\omega)$ should be $0^\circ$ (or a multiple of $360^\circ$). If $A_{vo}$ is a negative number, the phase shift through the amplifier is $\pm180^\circ$ and the phase through $\beta(j\omega)$ for positive feedback should be $\pm180^\circ \pm n360^\circ$. In other words, for positive feedback the total phase shift associated with the closed loop must be $0^\circ$ or a multiple $n$ of $360^\circ$.

From Figure 1.1 we can write

\[ v_o = A_v(j\omega)v_d \tag{1.1} \]

\[ v_f = \beta(j\omega)v_o \tag{1.2} \]

and

\[ v_d = v_i + v_f \tag{1.3} \]

Thus, from (1.1) to (1.3), the closed-loop voltage gain $A_{vf}(j\omega)$ is given by

\[ A_{vf}(j\omega) = \frac{v_o}{v_i} = \frac{A_v(j\omega)}{1 - \beta(j\omega)A_v(j\omega)} \tag{1.4} \]

The quantity $\beta(j\omega)A_v(j\omega)$ is known as the loop gain.
For oscillations to occur, an output signal must exist with no input signal applied. With $v_i = 0$ in (1.4) it follows that a finite $v_o$ is possible only when the denominator is zero. That is, when

$$1 - \beta(j\omega)A_v(j\omega) = 0$$

or

$$\beta(j\omega)A_v(j\omega) = 1 \quad (1.5)$$

Equation (1.5) expresses the fact that for oscillations to occur the loop gain must be unity. This relation is known as the Barkhausen criterion.

With $A_v(j\omega) = A_vo$ and letting

$$\beta(j\omega) = \beta_r(\omega) + j\beta_i(\omega)$$

where $\beta_r(\omega)$ and $\beta_i(\omega)$ are the real and imaginary parts of $\beta(j\omega)$, we can express (1.5) in the form

$$\beta_r(\omega)A_vo + j\beta_i(\omega)A_vo = 1$$

Equating the real and imaginary parts on both sides of the equation gives

$$\beta_r(\omega)A_vo = 1 \Rightarrow A_vo = \frac{1}{\beta_r(\omega)} \quad (1.6)$$

and

$$\beta_i(\omega)A_vo = 0 \Rightarrow \beta_i(\omega) = 0 \quad (1.7)$$

since $A_vo \neq 0$. The conditions in (1.6) and (1.7) are known as the Barkhausen criteria in rectangular form for $A_v(j\omega) = A_vo$.

The condition (1.6) is known as the gain condition, and (1.7) as the frequency of oscillation condition. The frequency of oscillation condition predicts the frequency at which the phase shift around the closed loop is $0^\circ$ or a multiple of $360^\circ$.

The relation (1.5) can also be expressed in polar form as

$$\beta(j\omega)A_v(j\omega) = |\beta(j\omega)A_v(j\omega)| \left| \Re(\beta(j\omega)A_v(j\omega)) \right| = 1$$

Hence, it follows that

$$|\beta(j\omega)A_v(j\omega)| = 1 \quad (1.8)$$

and

$$|\beta(j\omega)A_v(j\omega)| = \pm n360^\circ \quad (1.9)$$
where \( n = 0, 1, 2, \ldots \). Equation (1.9) expresses the fact that the signal must travel through the closed loop with a phase shift of \( 0^\circ \) or a multiple of \( 360^\circ \). For \( A_v(j\omega) = A_{vo} \), then \( |\beta(j\omega)A_{vo}| \) is the angle of \( \beta(j\omega) \), and the condition (1.9) is equivalent to saying that \( \beta_i(j\omega) = 0 \), in agreement with (1.7). Also, for \( A_v(j\omega) = A_{vo} \) and with \( \beta_i(j\omega) = 0 \), (1.8) reduces to (1.6). The conditions in (1.8) and (1.9) are known as the Barkhausen criteria in polar form.

When the amplifier is a current amplifier, the basic feedback network can be represented as shown in Figure 1.2. In this case, \( A_i(j\omega) \) is the current gain of the amplifier, and the current feedback factor \( \alpha(j\omega) \) is

\[
\alpha(j\omega) = \frac{i_f}{i_o}
\]

For this network, the condition for oscillation is given by

\[
\alpha(j\omega)A_i(j\omega) = 1
\]  

(1.10)

which expresses the fact that loop gain in Figure 1.2 must be unity.

The loop gain can be evaluated in different ways. One method that can be used in some oscillator configurations is to determine \( A_v(j\omega) \) and \( \beta(j\omega) \) and to form the loop gain \( A_v(j\omega)\beta(j\omega) \). In many cases it is not easy to isolate \( A_v(j\omega) \) and \( \beta(j\omega) \) since they are interrelated. In such cases a method that can usually be implemented is to represent the oscillator circuit as a continuous and repetitive circuit. Hence, the loop gain is calculated as the gain from one part to the same part in the following circuit. An alternate analysis method is to replace the amplifier and feedback network in Figure 1.1 by their ac models and write the appropriate loop equations. The loop equations form a system of linear equations that can be solved for the closed-loop voltage gain, which can be expressed in the general form.

\[ i_d = i_i + i_f \]

\[ A_i(j\omega) \]

\[ \alpha(j\omega) \]

\[ i_o \]

\[ i_f \]

\[ i_i \]
1.2 Oscillation Conditions

\[ A_{\text{eff}}(j\omega) = \frac{v_o}{v_i} = \frac{N(j\omega)}{D(j\omega)} \] (1.11)

where \(N(j\omega)\) represents the numerator polynomial and \(D(j\omega)\) is the system determinant of the linear equations. In terms of (1.11) the conditions for oscillations are obtained by setting the system determinant equal to zero (i.e., \(D(j\omega) = 0\)). Setting \(D(j\omega) = 0\) results in two equations: one for the real part of \(D(j\omega)\) (which gives the gain condition), and one for the imaginary part of \(D(j\omega)\) (which gives the frequency of oscillation).

From circuit theory we know that oscillation occurs when a network has a pair of complex conjugate poles on the imaginary axis. However, in electronic oscillators the poles are not exactly on the imaginary axis because of the nonlinear nature of the loop gain. There are different nonlinear effects that control the pole location in an oscillator. One nonlinear mechanism is due to the saturation characteristics of the amplifier. A saturation-limited sinusoidal oscillator works as follows. To start the oscillation, the closed-loop gain in (1.4) must have a pair of complex-conjugate poles in the right-half plane. Then, due to the noise voltage generated by thermal vibrations in the network (which can be represented by a superposition of input noise signals \(v_{n}\)) or by the transient generated when the dc power supply is turned on, a growing sinusoidal output voltage appears. The characteristics of the growing sinusoidal signal are determined by the complex-conjugate poles in the right-half plane. As the amplitude of the induced oscillation increases, the amplitude-limiting capabilities of the amplifier (i.e., a reduction in gain) produce a change in the location of the poles. The changes are such that the complex-conjugate poles move towards the imaginary axis. However, the amplitude of the oscillation was increasing and this makes the complex poles to continue the movement toward the left-half plane. Once the poles move to the left-half plane the amplitude of the oscillation begins to decrease, moving the poles toward the right-half plane. The process of the poles moving between the left-half plane and the right-half plane repeats, and some steady-state oscillation occurs with a fundamental frequency, as well as harmonics. This is a nonlinear process where the fundamental frequency of oscillation and the harmonics are determined by the location of the poles. Although the poles are not on the imaginary axis, the Barkhausen criterion in (1.5) predicts fairly well the fundamental frequency of oscillation. It can be considered as providing the fundamental frequency of the oscillator based on some sort of average location for the poles.

The movement of the complex conjugate poles between the right-half plane and the left-half plane is easily seen in an oscillator designed with an amplitude limiting circuit that controls the gain of the amplifier and, therefore, the motion of the poles. An example to illustrate this effect is given in Example 1.6.

The previous discussion shows that for oscillations to start the circuit must be unstable (i.e., the circuit must have a pair of complex-conjugate poles in the right-half plane). The condition (1.5) does not predict if the circuit is unstable. However, if the circuit begins to oscillate, the Barkhausen criterion in (1.5) can be used to predict the approximate fundamental frequency of oscillation and the gain condition. The stability of the oscillator closed-loop gain can be determined using the Nyquist stability test.
1.3 Nyquist Stability Test

There are several methods for testing the stability of a feedback amplifier. In general, (1.4) can be expressed in the form

\[ A_{vf}(s) = \frac{V_o}{V_i} = \frac{A_v(s)}{1 - \beta(s)A_v(s)} \] (1.12)

The stability \( A_{vf}(s) \) is determined by the zeroes of \( 1 - \beta(s)A_v(s) \) provided there is no cancellation of right-half plane poles and zeroes when forming the product \( \beta(s)A_v(s) \). In practical oscillators the previous pole-zero cancellation problems are unlikely to occur. If there are no pole-zero cancellation problems, the poles of \( A_v(s) \) are common to those of \( \beta(s)A_v(s) \) and of \( 1 - \beta(s)A_v(s) \). Therefore, the feedback amplifier is stable if the zeroes of \( 1 - \beta(s)A_v(s) \) lie in the left-half plane. In what follows we assume that there are no pole-zero cancellation problems.

The Nyquist stability test (or criterion) can be used to determine the right-half plane zeroes of \( 1 - \beta(s)A_v(s) \). A Nyquist plot is a polar plot of the loop gain \( \beta(s)A_v(s) \) for \( s = j\omega \) as the frequency \( \omega \) varies from \(-\infty < \omega < \infty \). Two typical Nyquist plots are shown in Figure 1.3. The Nyquist test states that the number of times that the loop-gain contour encircles the point \( 1 + j0 \) in a clockwise direction is equal to the difference between the number of zeroes and the number of poles of \( 1 - \beta(s)A_v(s) \) with positive real parts (i.e., in the right-half plane). The point \( 1 + j0 \) is called the critical point. To be specific, let \( N \) be the number of clockwise encirclements of the critical point by the Nyquist plot, let \( P \) be the number of right-half plane poles of \( \beta(s)A_v(s) \) (which are the same as those of \( 1 - \beta(s)A_v(s) \)), and let \( Z \) be the number of right-half plane zeroes of \( 1 - \beta(s)A_v(s) \). The Nyquist stability test states that \( N = Z - P \) (or \( Z = N + P \)). If \( Z > 0 \) (or \( N + P > 0 \)) the feedback amplifier is unstable and will oscillate under proper conditions. (Note: In the case that there is a right-half plane pole-zero cancellation, the Nyquist test is not sufficient to determine stability.)

Figure 1.3 (a) A Nyquist plot of a stable feedback amplifier and (b) a Nyquist plot of an unstable feedback amplifier.
If \( \beta(s)A_v(s) \) has no poles in the right-half plane, then it follows that \( 1 - \beta(s)A_v(s) \) has no poles in the right-half plane (i.e., \( P = 0 \)). Thus, in this case \( A_{\text{ef}}(s) \) is unstable (i.e., has right-half plane poles) only if \( 1 - \beta(s)A_v(s) \) has right-half plane zeroes (i.e., if \( N > 0 \)). In other words, for \( P = 0 \) the feedback amplifier is unstable when \( N > 0 \) (since \( N = Z \) when \( P = 0 \)). When \( \beta(s)A_v(s) \) is stable, the Nyquist test simply requires that the plot of \( \beta(s)A_v(s) \) as a function of \( \omega \) does not encircle the critical point for the feedback amplifier to be stable. An alternative way of stating the Nyquist test when \( \beta(s)A_v(s) \) is stable is: “If \( \beta(s)A_v(s) \) is stable, the feedback amplifier is stable if \( |\beta(j\omega)A_v(j\omega)| < 1 \) when the phase of \( \beta(j\omega)A_v(j\omega) \) is \( 0^\circ \) or a multiple of \( 360^\circ \).” This condition ensures that the critical point is not enclosed.

In the case that \( \beta(s)A_v(s) \) has a pole on the \( j\omega \) axis, the contour in the \( s \) plane must be modified to avoid the pole. For example, if the pole is at \( s = 0 \), the path moves from \( s = -j\infty \) to \( s = j0^- \) then from \( s = j0^- \) to \( s = j0^+ \) around a semicircle of radius \( \epsilon \) (where \( \epsilon \) approaches zero), and then from \( s = j0^+ \) to \( s = j\infty \). From \( s = j\infty \) the contour follows a semicircle with infinite radius and moves back to \( s = -j\infty \). Hence, the contour encloses all poles and zeroes that \( \beta(s)A_v(s) \) has in the right-half plane.

Two typical Nyquist plots for a feedback amplifier with a stable loop gain are shown in Figure 1.3. The solid curve corresponds to \( \omega \geq 0 \), and the dashed curve to \( \omega \leq 0 \). Since \( \beta(j\omega)A_v(j\omega) = [\beta(j\omega)A_v(j\omega)]^* \) it follows that the dashed curve is simply the mirror image of the solid curve. In Figure 1.3(a) the Nyquist plot does not enclose the critical point. It is seen that at the frequency \( \omega_x \) the phase of \( \beta(j\omega)A_v(j\omega) \) is \( 0^\circ \) and its magnitude is less than one. Hence, the amplifier associated with this Nyquist plot is stable. A typical Nyquist plot for an unstable feedback amplifier (with a stable \( \beta(s)A_v(s) \)) is shown in Figure 1.3(b). For this plot \( N = Z = 1 \), and the closed-loop response has one pole in the right-half plane.

**Example 1.1**

(a) Let \( \beta(s) = \beta_o \) be a real number and

\[
A_v(s) = \frac{K}{s(s + 1)(s + 2)}
\]

Hence,

\[
\beta(s)A_v(s) = \frac{\beta_o K}{s(s + 1)(s + 2)}
\]

and it follows that the number of poles of the loop gain in the right-half plane is zero (i.e., \( P = 0 \)). Therefore, the system is stable if the Nyquist plot of \( \beta(s)A_v(s) \) does not encircle the point \( 1 + j0 \) (i.e., if \( N = Z = 0 \)).

The Nyquist plot of \( \beta(s)A_v(s) \) for \( \beta_o K = 3 \) is shown in Figure 1.4(a). This plot shows that the system is stable since there are no encirclements of the \( 1 + j0 \) point.
The resulting Nyquist plot for $\beta_o K = 9$ is shown in Figure 1.4(b). In this case, the plot of $\beta(s)A_v(s)$ encircles the $1 + j0$ point twice in the clockwise direction. Hence, $N = Z = 2$, and the closed loop system is unstable because of two poles in the right-half plane.

In this part of the example the stability depended on the value of $\beta_o K$.

(b) Let $\beta(s) = \beta_o$ be a real number and

$$A_v(s) = \frac{K}{s(s + 1)(s - 1)}$$

Hence,

$$\beta(s)A_v(s) = \frac{\beta_o K}{s(s + 1)(s - 1)}$$

and it follows that $P = 1$, since there is a pole at $s = 1$. The Nyquist plots of the loop gain for $\beta_o K > 0$ and $\beta_o K < 0$ are shown in Figure 1.5. The solid curve in the plot corresponds to the mapping for $\omega > 0$, and the dashed curve for $\omega < 0$. Figure 1.5(a) shows that $N = 0$ when $\beta_o K > 0$, and Figure 1.5(b) shows that $N = 1$ when $\beta_o K < 0$; hence, the information in Table 1.1.

That is, the function $1 - \beta(s)A_v(s)$ for $\beta_o K > 0$ has a zero in the right-half plane, and for $\beta_o K < 0$ it has two zeroes in the right-half plane. Obviously, this feedback system is unstable for any real value of $\beta_o K$.

The information displayed in the polar Nyquist diagram can also be shown using Bode plots. Thus, the stability of an amplifier can also be determined from the Bode magnitude and phase plots of the loop gain. In terms of the magnitude and phase Bode plots of a stable $\beta(j\omega)A_v(j\omega)$, it follows that the closed-loop gain.
1.3 Nyquist Stability Test

Figure 1.5 Nyquist plots for Example 1.1(b) when (a) $\beta_0 K > 0$ and (b) $\beta_0 K < 0$.

Table 1.1 Values of $Z$ for Example 1.1(b)

<table>
<thead>
<tr>
<th>$\beta_0 K$</th>
<th>$P$</th>
<th>$N$</th>
<th>$Z = N + P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0 K &gt; 0$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_0 K &lt; 0$</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

is stable if $|\beta(j\omega)A_v(j\omega)|$ in dBs is smaller than 0 dB when the phase shift is $0^\circ$ (or a multiple of $360^\circ$). In other words, the plot of $|\beta(j\omega)A_v(j\omega)|$ in dBs crosses the 0-dB axis at a frequency lower than the frequency at which the phase reaches $0^\circ$ (or $\pm n360^\circ$). Typical Bode plots of the magnitude and phase of a stable feedback amplifier are shown in Figure 1.6.

Two important quantities in the determination of stability are the gain margin and the phase margin (shown in Figure 1.6). The gain margin is the number of decibels that $|\beta(j\omega)A_v(j\omega)|$ is below 0 dB at the frequency where the phase is $0^\circ$. The phase margin is the number of degrees that the phase is above $0^\circ$ at the frequency where $|\beta(j\omega)A_v(j\omega)|$ is 0 dB. A positive gain margin shows that the amplifier is potentially unstable. Similarly, a positive phase margin is associated with a stable amplifier. Of course, the gain margin and phase margin can also be shown in a Nyquist diagram.

Typical Bode plots of $\beta(j\omega)A_v(j\omega)$ for feedback amplifiers having one, two, and three poles with $\beta(0)A_v(0) = -K < 0$ are shown in Figure 1.7. The single-pole loop-gain function shown in Figure 1.7(a) has a minimum phase shift of $90^\circ$. Therefore, this amplifier is always stable. Figure 1.7(b) shows a loop gain having two poles. Again this amplifier is always stable because the phase shift is positive and approaches $0^\circ$ only at $\omega = \infty$. Figure 1.7(c) shows a three-pole loop gain that is stable since $|\beta(j\omega)A_v(j\omega)|$ is below 0 dB at the frequency where the phase is $0^\circ$ (i.e., the gain margin is negative). Figure 1.7(d) shows a three-pole loop gain that is unstable, since the phase is less than $0^\circ$ at the frequency where $|\beta(j\omega)A_v(j\omega)|$ is 0 dB (i.e., the phase margin is negative).

It is of interest to see how the Nyquist and Bode plots portray the stability information and their relation to the closed loop and transient responses of the
feedback amplifier. This is illustrated in Figure 1.8. In the Nyquist plots only the positive frequencies are shown. In the Bode plot the solid curve is for the magnitude of the closed loop response, and the dashed curve is for the phase. Figure 1.8(a) illustrates a stable feedback amplifier with a large positive phase margin. Observe the Bode plots, $|A_{v_f}(j\omega)|$, and the transient response. In Figure 1.8(a), as well as in the other figures, the frequency at which $|\beta(j\omega)A_v(j\omega)| = 1$ is $f_1$, and the frequency at which $|\beta(j\omega)A_v(j\omega)| = 0^\circ$ is $f_2$. The phase margin in Figure 1.8(a) is positive. Figure 1.8(b) illustrates a stable feedback amplifier with a smaller positive phase margin. Observe the larger peak in the associated $|A_{v_f}(j\omega)|$ response and in the transient response.

Figure 1.8(c) illustrates an ideal oscillator. The oscillation conditions are satisfied, since $|\beta(j\omega)A_v(j\omega)| = 1$ and $|\beta(j\omega)A_v(j\omega)| = 0^\circ$ at $f = f_1 = f_2$, which results in an ideal stable sinusoidal oscillation (see the plot of $v_o(t)$). Figure 1.8(d) illustrates an unstable oscillation. Observe that $|\beta(j\omega)A_v(j\omega)| > 1$ when $|\beta(j\omega)A_v(j\omega)| = 0^\circ$; hence, positive feedback occurs and $v_o(t)$ shows the associated growing sinusoidal response. Basically, Figure 1.8(c) shows what happens when the complex poles move to the imaginary axis, and Figure 1.8(d) shows what happens when the complex conjugate poles remain in the right-half plane. As we will see, there are ways to determine if the oscillation will be stable or not.

1.4 Root Locus

A root-locus plot is a convenient method to analyze the motion of the closed-loop gain poles in the complex $s$ plane as a function of the amplifier gain, or as a function of the feedback factor. In order to use this method, the denominator of
Figure 1.7  Bode plots for a loop gain having (a) one pole, (b) two poles, (c) three poles (stable case), and (d) three poles (unstable case).
Figure 1.7 (Continued).
Figure 1.8 Nyquist, Bode, $|A_{v f}(j\omega)|$, and transient response plots of (a) a stable feedback amplifier with a large phase margin, (b) a stable feedback amplifier with a smaller phase margin, (c) a stable oscillator, and (d) an unstable oscillator.

$A_{v f}(s)$ is expressed in polynomial form. The stability of the feedback amplifier is analyzed by observing how the poles of $A_{v f}(s)$ move in the $s$ plane. A typical analysis consists in studying the motion of the roots of $A_{v f}(s)$ as a function of the amplifier open-loop gain, and determining the value of gain that move the roots to the imaginary axis at $s = -j\omega_0$. The value of gain and the frequency $\omega_0$ are identical to the values predicted by the Barkhausen criterion (i.e., the gain condition and the frequency of oscillation condition).

Consider the root-locus analysis of a feedback amplifier with a two-pole $A_{v}(s)$ given by

$$A_{v}(s) = \frac{A_o}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$
and a constant feedback $\beta_o$, where $\beta_o < 0$, and $A_o$ is the midband value of $A_v(s)$. Hence, $\beta_o A_o < 0$ and

$$\beta(s)A_v(s) = \frac{\beta_o A_o}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

The closed-loop gain of the feedback amplifier is given by

$$A_{vf}(s) = \frac{A_v(s)}{1 - \beta_o A_v(s)} = \frac{A_o \omega_1 \omega_2}{s^2 + s(\omega_1 + \omega_2) + \omega_1 \omega_2(1 - \beta_o A_o)} \tag{1.13}$$

The root locus of the poles in (1.13) follows from the analysis of

$$s^2 + s(\omega_1 + \omega_2) + \omega_1 \omega_2(1 - \beta_o A_o) = 0$$

as $A_o$ varies. The root locus is shown in Figure 1.9. This plot shows that for $A_o$ approaching zero the roots are located at $s_1 = -\omega_1$ and $s_2 = -\omega_2$. As $A_o$ increases, the roots move along the negative real axis as shown in Figure 1.9. At a specific value of $A_o$, denoted by the value of $A_o = A'_o$ (see Figure 1.9), the roots are identical, and for $A_o > A'_o$ the roots become complex but remain in the left-half plane. Therefore, this feedback amplifier is stable. The value of $A'_o$ is given by

$$A'_o = \frac{1}{\beta_o} \left[1 - \frac{0.25(\omega_1 + \omega_2)^2}{\omega_1 \omega_2}\right]$$

Of course, if $\beta_o A_o > 0$ the feedback amplifier is unstable.

Figure 1.9 Root locus of a two-pole function $A_v(s)$. 
Next, consider a three-pole $A_v(s)$ given by

$$A_v(s) = \frac{A_o}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)\left(1 + \frac{s}{\omega_3}\right)}$$

with a constant feedback $\beta_o$, where $\beta_o < 0$ (i.e., $\beta_o A_o < 0$). In this case $A_{vf}(s)$ is given by

$$A_{vf}(s) = \frac{A_o}{a_3 s^3 + a_2 s^2 + a_1 s + (1 - \beta_o A_o)}$$

where

$$a_3 = \frac{1}{\omega_1 \omega_2 \omega_3}$$

$$a_2 = \frac{1}{\omega_1 \omega_2} + \frac{1}{\omega_1 \omega_3} + \frac{1}{\omega_2 \omega_3}$$

and

$$a_1 = \frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3}$$

The root locus of

$$a_3 s^3 + a_2 s^2 + a_1 s + (1 - \beta_o A_o) = 0$$

is shown in Figure 1.10. For $A_o$ approaching 0, the poles are located at $s_1 = -\omega_1$, $s_2 = -\omega_2$, and $s_3 = -\omega_3$. As $A_o$ increases, the pole $s_3$ moves along the negative real axis towards $-\infty$, and the poles $s_1$ and $s_2$ become complex conjugate poles. At a certain value of $A_o$ the poles are located on the imaginary axis at $s_{1,2} = \pm j\omega_0$ and oscillation occurs. Figure 1.10 also shows that certain values of $A_o$ move the poles into the right-half plane.

Another open-loop gain function that can lead to oscillations is

$$A_v(s) = \frac{A_o \left(1 - \frac{s}{\omega_3}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

This function has two poles and a right-half plane zero at $s = \omega_3$. This type of transfer function occurs in the high-frequency analysis of several amplifier configurations.
Example 1.2

Consider the open-loop gain of a CE amplifier given by

\[ A_v(s) = \frac{A_o \left(1 - \frac{s}{\omega_3}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)} \]

where \(A_o = 2,200\), \(\omega_1 = 10^6\) rad/s, \(\omega_2 = 10^8\) rad/s, and \(\omega_3 = 10^9\) rad/s. Assume that the feedback factor is constant and given by \(\beta(j\omega) = \beta_o = -0.1\). The loop gain is

\[ \beta(j\omega) A_v(j\omega) = \frac{-0.1(2,200)\left(1 - \frac{s}{10^9}\right)}{\left(1 + \frac{s}{10^6}\right)\left(1 + \frac{s}{10^8}\right)} \quad (1.14) \]

The Nyquist plot of (1.14) is shown in Figure 1.11(a). The behavior of the function in (1.14) around the critical point is difficult to see in Figure 1.11(a). A graph showing the behavior around the critical point is shown in Figure 1.11(b). From (1.14) it follows that \(P = 0\), and from the Nyquist plots in Figure 1.11(a, b) we obtain \(N = 0\), since the critical point is not enclosed. Therefore, the feedback amplifier with \(A_o = 2,200\) is stable, since \(Z = N + P = 0\).

Next, assume that \(A_o\) is given by \(A_o = 22,000\). The Nyquist plot for this case is shown in Figure 1.11(c) and the behavior around the critical point in Figure...
1.4 Root Locus

Figure 1.11  (a) Nyquist plot for Example 1.2 with $A_o = 2,220$, (b) the behavior of the loop gain (with $A_o = 2,220$) around the critical point, (c) Nyquist plot for Example 1.2 with $A_o = 22,000$, and (d) the behavior of the loop gain (with $A_o = 22,000$) around the critical point.

1.11(d). Figure 1.11(d) shows that the critical point is enclosed twice, or $N = 2$. Hence, there are two roots in the right-half plane and the amplifier is unstable with $A_o = 22,000$.

The root locus of $A_{vf}(s)$ for this example follows from the analysis of

$$s^2 + s\left(\omega_2 + \omega_1 + \frac{\omega_1 \omega_2 \beta_o A_o}{\omega_3}\right) + \omega_1 \omega_2 (1 - \beta_o A_o) = 0$$

or

$$s^2 + s(101 \times 10^6 - 10^4 A_o) + 10^{14}(1 + 0.1 A_o) = 0 \quad (1.15)$$

The resulting root-locus plot is shown in Figure 1.12. For $A_o$ approaching 0 the roots are located at $s_1 = -10^6$ and $s_2 = -10^8$. At the value of $A_o = 10,100$ the complex poles are on the imaginary axis at $s_1 = j318 \times 10^6$ and $s_2 = -j318 \times 10^6$, and oscillations occur.
1.5 Routh-Hurwitz Method

Another method that can be used to determine if a polynomial has right-half plane roots is the Routh-Hurwitz method. In the Routh-Hurwitz method the denominator polynomial of $A_{vf}(s)$ is written in the form

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0$$

The coefficients are then arranged in the following array form (known as the Routh-Hurwitz array):

$$
\begin{array}{c|cccc}
    s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
    s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
    s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
    s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \ldots \\
    & & & & \\
    s^1 & & & & \\
    s^0 & & & & \\
\end{array}
$$

which shows that a polynomial of degree $n$ has $n + 1$ rows in the Routh-Hurwitz array. The terms $b_{n-1}, b_{n-3}, b_{n-5},$ etc., are the first, second, third, etc., entries in the $(n - 2)$ row. The terms $c_{n-1}, c_{n-3}, c_{n-5},$ etc., are the first, second, third, etc., entries in the $(n - 3)$ row. The entries are defined by
The entries in the first column (i.e., \(a_n, a_{n-1}, b_{n-1}, c_{n-1}\), etc.) are called the leading entries in the array. If none of the leading entries vanish, the number of roots of \(D(s)\) in the right-half plane is equal to the number of sign changes in the leading entries.

Example 1.3
Determine if the following denominator polynomials of \(A_{v_f}(s)\) have right-half plane roots:

(a) \(D(s) = s^4 + 10s^3 + 35s^2 + 50s + 24\) \hspace{1cm} (1.16)

(b) \(D(s) = s^3 + s^2 + 2s + 24\) \hspace{1cm} (1.17)

Solution
(a) The Routh-Hurwitz array for (1.16) is shown in Figure 1.13(a). The leading entries (i.e., 1, 10, 30, 42, and 24) are positive. Hence, \(D(s)\) in (1.16) has no right-half plane roots, or \(A_{v_f}(s)\) has no right-half plane poles.

The polynomial in (1.16) can be shown to be equal to

\[
D(s) = (s + 1)(s + 2)(s + 3)(s + 4)
\]

which obviously has no right-half plane roots.

(b) The Routh-Hurwitz array for the polynomial in (1.17) is shown in Figure 1.13(b). In this case, the leading entries have two sign changes. Hence, the polynomial in (1.17) has two right-half plane roots, and therefore, \(A_{v_f}(s)\) is unstable. In fact, the polynomial in (1.17) can be shown to be equal to

\[
D(s) = (s + 2)(s^2 + 1)\]

\[
D(s) = (s^2 + 1)(s + 2)
\]
\[ D(s) = (s + 3)(s - 1 + j2.6458)(s - 1 - j2.6458) \]

which has two roots in the right-half plane.

The Routh-Hurwitz method can also be used to determine at what value of the open-loop gain or feedback factor value is the closed-loop gain unstable. Example 1.4 illustrates this point.

Example 1.4
In an oscillator, the denominator polynomial of \( A_{vf}(s) \) is

\[ D(s) = s^2 + (3 - A_o)s + \omega_o^2 \]

where \( A_o \) is the amplifier gain and \( \omega_o \) is the frequency of oscillation. Determine for what values of \( A_o \) there are right-half plane roots.

**Solution**
The Routh-Hurwitz array for (1.18) is shown in Figure 1.14.

The leading entries are 1, 3 – \( A_o \), and 2. Hence, the feedback system is stable if 3 – \( A_o \) > 0 or \( A_o \) < 3. The feedback system is unstable when 3 – \( A_o \) < 0 or \( A_o \) > 3. Oscillations can occur when \( A_o = 3 \). This oscillator is analyzed in detail in the next section.

There are some degenerate cases that can occur in the Routh-Hurwitz array. One case occurs when a leading entry vanishes, and at least one entry in the corresponding row is nonzero. The other case occurs when a complete row vanishes. For these degenerate cases the reader is referred to an appropriate textbook in control systems.

Next, the loop-gain associated with an oscillator is analyzed. The oscillator selected is the Wien bridge, which provides an example where the feedback factor and the open-loop voltage gain are evaluated separately. In many oscillators the feedback network is loaded by the amplifier, and this effect must be taken into consideration in the analysis of the loop gain.

### 1.6 The Wien-Bridge Oscillator

The Wien-bridge oscillator is shown in Figure 1.15. The four arms of the bridge are \( R_1, R_2, Z_a, \) and \( Z_b \). The op amp maintains the voltage across two of the arms equal, since \( v_- = v_+ \). This oscillator can be used to analyze in closed form the conditions for oscillation, its stability, and the location of the complex poles.

\[
\begin{array}{c|cc}
\text{ } & 1 & 2 \\
\text{s^2} & 3 - A_o \\
\text{s} & 2 \\
\end{array}
\]

**Figure 1.14** Routh-Hurwitz array for Example 1.4.
From Figure 1.15 it is seen that there are two feedback paths. Positive feedback occurs between \( v_o \) and \( v_+ \) through the voltage divider formed by \( Z_a \) and \( Z_b \), and negative feedback occurs between \( v_o \) and \( v_- \) through \( R_1 \) and \( R_2 \).

The Wien-bridge oscillator in Figure 1.15 uses an op amp in an inverting configuration (i.e., the negative feedback path) to provide the open-loop gain. The open-loop gain is constant and given by

\[
A_v(j\omega) = A_{v0} = \frac{v_o}{v_+} = 1 + \frac{R_2}{R_1} \tag{1.19}
\]

The voltage \( v_+ \) is given by

\[
v_+ = v_o \frac{Z_b}{Z_b + Z_a} = v_o \frac{R}{1 + j\omega RC} \div \left( R + \frac{1}{j\omega C} + \frac{R}{1 + j\omega RC} \right) = v_o \frac{1}{3 + j \left( \omega RC - \frac{1}{\omega RC} \right)}
\]

Therefore, the voltage-feedback transfer function \( \beta(j\omega) \) can be written in the form

\[
\beta(j\omega) = \frac{v_+}{v_o} = \frac{1}{3 + j \left( \omega RC - \frac{1}{\omega RC} \right)} \tag{1.20}
\]

From (1.19) and (1.20) it follows that the loop gain is
For oscillation the loop gain must be unity. The frequency of oscillation and the gain condition are given by (1.7) and (1.6), respectively.

The condition (1.7) requires that the imaginary part of $\beta(j\omega)$ be zero. From (1.20) it is seen that $\beta_i(j\omega) = 0$ when

$$\omega RC - \frac{1}{\omega RC} = 0 \Rightarrow \omega = \omega_o = \frac{1}{RC}$$

or

$$f_o = \frac{1}{2\pi RC}$$

The frequency $f_o$ is the frequency of oscillation.

At $\omega = \omega_o$ the real part of $\beta(j\omega)$ is

$$\beta_r(\omega_o) = \frac{1}{3}$$

Therefore, from (1.6) the gain condition is

$$A_{vo} = \frac{1}{\beta_r(\omega_o)} = 3$$

and from (1.19) it follows that $R_2 = 2R_1$. In practice, the gain should be greater than 3 to start the oscillation. Values of 3.1 to 3.3 are reasonable.

It is interesting to observe that the op amp uses negative feedback through $R_1$ and $R_2$ to provide the required gain $A_{vo}$, and it uses positive feedback through the $RC$ networks to obtain the required closed-loop phase shift. The attenuation in the positive feedback loop must be equal to the gain $A_{vo}$, making the loop gain equal to one. Since the phase shift through the op amp is zero at $\omega_o$, the phase shift through the $RC$ networks must also be zero so the feedback signal $v_+$ is in phase with $v_o$ [see (1.20) at $\omega = \omega_o$].

The frequency of oscillation can be varied by simultaneously changing the capacitance values using a ganged capacitor arrangement. In addition, different frequency ranges can be selected by simultaneously switching different values of the resistors $R$.

A final observation is that op amps in this book are assumed to operate from a dual power supply. Of course, op amps with a single power supply can be used if proper single-supply techniques are used.

Further insight into the Wien-bridge oscillator is obtained by analyzing the oscillator in terms of the circuit poles. The poles of the closed-loop gain $A_{vf}(s)$ are the roots of $1 - \beta(s)A_{vo} = 0$. Using (1.21) with $s = j\omega$ the roots of $A_{vf}(s)$ are given by

$$\beta(j\omega)A_{vo} = \frac{1}{3 + j(\omega RC - \frac{1}{\omega RC})\left(1 + \frac{R_2}{R_1}\right)}$$

(1.21)
The Wien-Bridge Oscillator

1.6

The Wien-Bridge Oscillator

\[
1 - \frac{A_{v0}}{3 + \frac{s}{\omega_o} + \frac{\omega_o}{s}} = 0
\]

or

\[
s^2 + (3 - A_{v0}) \omega_o s + \omega_o^2 = 0
\] (1.23)

The two roots of (1.23) are shown in Figure 1.16 in a root-locus plot as a function of \( A_{v0} \). As \( A_{v0} \) varies from 0 to 1, the poles move along the negative real axis. For \( A_{v0} = 1 \), the poles meet at \(-\omega_o\). As \( A_{v0} \) varies from 1 to 3, the poles are complex and move in a semicircular path towards the imaginary axis. For \( A_{v0} = 3 \), the complex poles are located at \( j\omega_o \) and \(-j\omega_o\), respectively. As \( A_{v0} \) increases above 3, the complex poles move into the right-half plane; and at \( A_{v0} = 5 \) the poles meet on the positive real axis at \( \omega_o \). For \( A_{v0} > 5 \), they move along the positive real axis.

The conditions that produce complex poles are now analyzed. Observe that \( 1 < A_{v0} < 3 \) corresponds to a loop gain of \( \beta(j\omega_o)A_{v0} < 1 \), and \( 3 < A_{v0} < 5 \) corresponds to a loop gain of \( \beta(j\omega_o)A_{v0} > 1 \). For \( 1 < A_{v0} < 3 \) the poles produce an output voltage with an exponentially damped sinusoidal response, and for \( 3 < A_{v0} < 5 \) an exponentially growing sinusoidal response results. In order to start the oscillation, a value of gain slightly greater than 3 is used. Thus, at the start of oscillation the complex poles are in the right-half plane and an exponentially growing sinusoidal oscillation is produced. As the amplitude of the oscillation increases, the op amp saturates and its gain decreases. When the gain is 3, the poles are on the imaginary axis (i.e., to \( s = \pm j\omega_o \)); and when the gain goes below 3, the poles move into the left-half plane. In the Wien-bridge oscillator in Figure 1.15, the amplitude of the sinusoidal oscillation is limited by the saturation of the op amp. Hence, the oscillation amplitude varies between approximately \( V^+ - 1 \) and \( V^- + 1 \).

Wien-bridge oscillators work very well for frequencies up to about 1 MHz. With an amplitude-limiting circuit the harmonic distortion can be less than 5%.

The Wien bridge can also be constructed with different values of the branch values of \( R \) and \( C \). If

![Figure 1.16](image)

**Figure 1.16** Poles of the closed-loop gain as a function of \( A_{v0} \).
Theory of Oscillators

\[ Z_a = R_a + \frac{1}{j\omega C_a} \]

and

\[ Z_b = \frac{R_b}{1 + j\omega C_b} \]

it follows that for oscillation: \( R_a C_a = R_b C_b \), and the frequency of oscillation is given by

\[ f_o = \frac{1}{2\pi R_a C_a} = \frac{1}{2\pi R_b C_b} \]

The required gain of the amplifier is

\[ A_{vo} = \frac{1}{\beta(j\omega_o)} = 1 + \frac{2R_1}{R_2} \]

Example 1.5
Design the Wien-bridge oscillator shown in Figure 1.15 to oscillate at 5 kHz.

Solution
A practical value of 0.01 \( \mu \)F for the capacitors can be selected. Then, using (1.22), the value of \( R \) is

\[ R = \frac{1}{2\pi f_o C} = \frac{1}{2\pi(5 \times 10^3)(0.01 \times 10^{-6})} = 3.18 \text{ k}\Omega \]

A practical value of 3 k\Omega can be used in series with a trimming potentiometer to set the frequency of oscillation at 5 kHz.

To start the oscillation, a value of \( A_{vo} = 3.2 \) is used. From (1.19), a gain of 3.2 is obtained with \( R_2 = 22 \text{ k}\Omega \) and \( R_1 = 10 \text{ k}\Omega \). The supply voltages of the op amp can be selected as 12V and -12V. The transient simulation of the oscillator, using a 741 op amp is shown in Figure 1.17. Observe that the output voltage reaches saturation producing a clipping in the output waveform and, therefore, a significant amount of distortion. This occurs because the starting condition requires \( A_{vo} > 3 \), and the gain of the amplifier changes when its output reaches saturation. The fundamental frequency of oscillation (i.e., \( f_o = \text{freq}[1] = 5.0002 \) kHz) is close to the predicted value using \( \beta(j\omega_o)A_{vo} = 1 \).

Example 1.5 shows that some sort of amplitude-limiting mechanism is needed to reduce the harmonic distortion. There are several ways of accomplishing amplitude limiting. The amplitude of oscillation is determined by the loop gain, which is made to be greater than 1 in order to start the oscillation, and by the nonlinearities
Figure 1.17  Simulation of the Wien-bridge oscillator in Example 1.5.
of the amplifier and feedback network. If the feedback network is a passive network, the nonlinearities of the amplifier determine the amplitude of oscillation. This procedure, as seen in Figure 1.17, generates unwanted harmonics since the amplitude of the oscillation is limited by the saturation voltage of the op amp. In order to remove these harmonics, a bandpass filter that passes only the oscillation frequency can be used after the amplifier. Of course, the nonlinearities of the amplifier can be avoided by limiting somehow the amplitude of the oscillation before the amplitude reaches the amplifier’s saturation value. There are many circuits that can be used to limit the amplitude of the oscillator. Some of these circuits are shown in Figure 1.18.

Figure 1.18(a) shows a limiting circuit using back-to-back Zener diodes (usually $V_{Z1} = V_{Z2}$). The output is limited to $- (V_{Z2} + 0.7) < v_o < (V_{Z1} + 0.7)$. When the Zeners are not conducting, the op-amp gain is $-R_2/R_1$. The transfer function of this circuit is illustrated in Figure 1.18(a). Another limiting circuit is shown in Figure 1.18(b) with its transfer function. In this circuit the gain between $v_o$ and $v_{IN}$ changes from $A_v1$ (when the Zeners conduct) to $A_v2$ (when the Zeners are not conducting).

An amplitude-limiting mechanism is basically an automatic gain control (AGC) circuit that forces the amplifier gain to decrease when the amplitude of the

Figure 1.18  (a) A limiting circuit and the associated transfer functions and (b) another limiting circuit.
oscillation increases. Figure 1.19 shows three Wien-bridge oscillators with amplitude-limiting mechanisms. In Figure 1.19(a), when the diodes are off, the gain is $1 + R_2 \parallel R_1$; and when a diode is on, the gain is reduced to $1 + (R_2 \parallel R_3)/R_1$. The start up condition requires a gain slightly greater than 3 or

$$\frac{R_2}{R_1} > 2$$  \hspace{1cm} (1.24)

and when a diode is on, the gain should be slightly less than 3, or

$$\frac{R_2 \parallel R_3}{R_1} < 2$$  \hspace{1cm} (1.25)

The inequality in (1.24) can be satisfied by making it equal to a value between 2.1 to 2.2 and in (1.25) using a value between 1.8 and 1.9.

When a diode is conducting, the amplitude of the output voltage is limited. Since $v_+ = v_- = v_o/3$, a nodal equation gives

$$v_o = \frac{v_o - \frac{v_o}{3}}{R_2} + \frac{v_o - \frac{v_o}{3} - V_D}{R_3}$$

![Figure 1.19](image-url)  \hspace{1cm} (a)

![Figure 1.19](image-url)  \hspace{1cm} (b)

![Figure 1.19](image-url)  \hspace{1cm} (c)

**Figure 1.19** Three Wien-bridge oscillators with (a, b, c) amplitude-limiting mechanism.
Theory of Oscillators

or

\[ v_o = \frac{3V_D}{2\left(1 + \frac{R_3}{R_2}\right) - \frac{R_3}{R_1}} \tag{1.26} \]

where \( V_D \approx 0.5V \) for an actual diode.

Equation (1.26) shows that the amplitude of \( v_o \) depends on \( R_3 \). This equation can be used to obtain the value of \( R_3 \) for a given amplitude of \( v_o \). Since in this oscillator the gain is controlled by the circuit, it provides an opportunity to verify that the location of the poles changes between the right-half plane and the left-half plane, and how the pole movement is related to the Barkhausen criterion in the determination of the frequency of oscillation (see Example 1.6).

In Figure 1.19(b) the amplitude-limiting mechanism is implemented with Zener diodes. The analysis of this oscillator is similar to the previous analysis. The output voltage is given by (1.26) with \( V_D \) replaced by \( V_Z + V_D \). A simple way of designing this oscillator is to let \( R_2 \parallel R_1 = 2.15 \) and \( (R_2 \parallel R_3)/R_1 = 1.8 \), which can be satisfied with \( R_2 = 2R_a, R_1 = 0.93R_a, \) and \( R_3 = 10R_a \) (\( R_a \) is a scaling factor). As an example, with \( V_Z = 4.2V \), and selecting \( R_a = 5 \text{k}\Omega \), it follows that \( R_1 = 4.65 \text{k}\Omega, R_2 = 10 \text{k}\Omega, \) and \( R_3 = 50 \text{k}\Omega \). The amplitude of the oscillation is limited to \( v_o = 9.7V \).

In Figure 1.19(c) a modification of the amplitude control circuit that produces smaller values of the control resistors is shown. For this circuit the design equations are

\[ R_2 + \frac{R_x}{R_1} > 2 \]

\[ \frac{(R_2 \parallel R_3) + R_x}{R_1} < 2 \]

and

\[ v_o = \frac{3V_D}{\left(2 - \frac{R_x}{R_1}\right)\left(1 + \frac{R_3}{R_2}\right) - \frac{R_3}{R_1}} \tag{1.27} \]

Obviously, as \( R_x \rightarrow 0 \), (1.27) reduces to (1.26).

Usually, in these oscillators a buffer stage (i.e., a unity gain amplifier) is used to couple the oscillation signal.

The following example helps to explain why, although not exact, the Barkhausen criterion is useful in predicting the expected frequency of oscillation. As Lindberg [1] mentioned, the placement of the poles on the imaginary axis is an impossible act of balance.
Example 1.6
(a) Design the Wien-bridge oscillator shown in Figure 1.20(a) to oscillate at $\omega_o = 10$ krad/s with an amplitude of 2V.
(b) Repeat part (a) for an amplitude of 10V.

Solution
(a) From (1.22) with $C = 0.01 \, \mu F$ and $R = 10 \, k\Omega$, the desired $\omega_o = 10$ krad/s is obtained.

To start the oscillation, a value of $A_{vo} = 3.2$ (or $R_2 || R_1 = 2.2$) is used. This can be obtained with $R_2 = 11 \, k\Omega$ and $R_1 = 5 \, k\Omega$. From (1.26), if $V_D = 0.45V$, an amplitude of 2V is obtained with $R_3 = 73 \, k\Omega$; and if $V_D = 0.6V$, the value is $R_3 = 60.5 \, k\Omega$.

![Figure 1.20](image)

(a) ADS simulation for Example 1.6 for $v_o = 2V$, (b) transient simulation results, and (c) simulation result for $v_o = 10V$. 
The harmonic-balance simulation of this oscillator is shown in Figure 1.20(a) with $R_3 = 73 \text{k}\Omega$, and the transient simulation data is shown in Figure 1.20(b). The fundamental frequency of oscillation is $\text{freq}[1] = 1.598 \text{ kHz}$ (or $\omega_0 = 10.04 \text{ krad/s}$). The current waveform in the diodes shows the on/off states of the diodes.

For this design, when the diodes are off the gain is $A_{vo} = 3.2$. From (1.23), the poles of the closed-loop gain are located at

\[ s^2 - 2,000s + 10^8 = 0 \]

or

\[ s_{1,2} = 1,000 \pm j9,949.9 \] \hspace{1cm} (1.28)

Therefore, the poles are in the right-half plane.

When a diode conducts, the gain is reduced to

\[ A_{vo} = 1 + \frac{73 \times 10^3 \| 11 \times 10^3}{5 \times 10^3} = 2.912 \]

and the new location of the poles, using (1.23), is

\[ s^2 + 880s + 10^8 = 0 \]

or

\[ s_{1,2} = -440 \pm j9,990.3 \] \hspace{1cm} (1.29)

From (1.28) and (1.29) it is seen that the poles move between the right-half plane and the left-half plane. The frequency associated with the poles in the right-half plane is $\omega_{o,1} = 9,949.9 \text{ rad/s}$, and the frequency associated with the poles in the left-half plane is $\omega_{o,2} = 9,990.3 \text{ rad/s}$. The frequency of oscillation predicted by the Barkhausen criterion is $\omega_0 = 10 \text{ krad/s}$, which occurs if the poles are on the $j\omega$ axis. However, the poles of this oscillator are moving between $\omega_{o,1}$ and $\omega_{o,2}$. This variation in frequency produces a fundamental frequency of oscillation and the associated harmonics.

(b) An amplitude of $v_o = 10\text{V}$ can be obtained with $R_1 = 5 \text{k}\Omega$, $R_2 = 11 \text{k}\Omega$, and $R_3 = 101 \text{k}\Omega$. When the diodes are not conducting, the poles are in the right-half plane and are given by (1.28). When a diode conducts the poles move to the left-half plane and $A_{vo} = 2.9821$. Their location is calculated using (1.23), namely,

\[ s^2 + 179s + 10^8 = 0 \]

or

\[ s_{1,2} = -89.5 \pm j9,979.6 \]
1.6 The Wien-Bridge Oscillator

Hence, the poles of this oscillator are moving between \( \omega_{o,1} = 9,949.9 \text{ rad/s} \) and \( \omega_{o,2} = 9,979.6 \text{ rad/s} \).

The simulation results are shown in Figure 1.20(c). The fundamental frequency of oscillation is 1.598 kHz.

Another Wien-bridge oscillator with an amplitude-limiting circuit is shown in Figure 1.21. In this oscillator the amplitude-limiting circuit consists of the diodes \( D_1 \) and \( D_2 \), and the resistors \( R_3 \), \( R_4 \), \( R_5 \), and \( R_6 \). To understand the operation of the amplitude-limiting circuit, observe that as \( v_o \) increases, the voltage at node \( v_y \) will exceed the voltage \( v_1 \), forcing \( D_2 \) to conduct. When \( D_2 \) conducts, the value of \( v_y \) is \( v_y = v_1 + 0.7 \), and \( v_o \) is clamped at the value \( v_o(\text{max}) \), given by

\[
  v_y = v_1 + 0.7 = \frac{v_o(\text{max}) R_6}{R_5 + R_6} + \frac{V^- R_5}{R_5 + R_6} \tag{1.30}
\]

Since \( v_1 \) is approximately \( v_o/3 \), it follows from (1.24) that

\[
  \frac{v_o(\text{max})}{3} + 0.7 = \frac{v_o(\text{max}) R_6}{R_5 + R_6} + \frac{V^- R_5}{R_5 + R_6} \tag{1.31}
\]

Similarly, as \( v_o \) decreases, the voltage \( v_x \) will drop below \( v_1 \), forcing \( D_1 \) to conduct. When \( D_1 \) conducts, the voltage \( v_x \) is \( v_x = v_1 - 0.7 \), and \( v_o \) is clamped at the value \( v_o(\text{min}) \), given by

\[\text{Figure 1.21} \quad \text{Wien-bridge oscillator with an amplitude-limiting circuit.}\]
The simultaneous solution of (1.31) and (1.32) gives the value of the resistors that limit the output voltage to $v_o(\text{min}) < v_o < v_o(\text{max})$. In order to obtain a symmetrical sinusoidal voltage, the selection $R_3 = R_6$ and $R_4 = R_5$ is usually made.

Example 1.7
Design an amplitude-limiting circuit for the 5-kHz Wien-bridge oscillator in Example 1.5. The amplitude of the sinusoidal output voltage is to be limited to $|v_o| = 5\text{V}$.

Solution
With $v_o(\text{max}) = 5\text{V}$ and $v_o(\text{min}) = -5\text{V}$, it follows from (1.31) and (1.32) that $R_3 = R_6 = 10.9\text{ k}\Omega$ and $R_4 = R_5 = 2\text{ k}\Omega$.

To start the oscillation a voltage gain of 3.2 is obtained with $R_2 = 22\text{ k}\Omega$ and $R_1 = 10\text{ k}\Omega$. The simulation of the oscillator is shown in Figure 1.22. Obviously, the harmonic content of the oscillator in Figure 1.22 is much less than the one in Figure 1.17.

Another way of stabilizing the amplitude of oscillation is to use a thermistor in the circuit. Thermistors with positive- and negative-temperature coefficients are available. A negative-temperature coefficient thermistor has a resistance that decreases as the temperature increases. For example, in Figure 1.15 a negative-temperature coefficient thermistor can be used for $R_2$. Hence, as the amplitude of oscillation increases, the current in $R_2$ increases and its resistance decreases, resulting in a decrease in $A_{v_0}$ which makes $\beta(j\omega_o)A_{v_0} = 1$.

A positive-temperature coefficient device that can also be used to stabilize the oscillation is a tungsten filament lamp. In low-wattage lamps, cold resistances of a few ohms to hundreds of ohms are available, with hot resistances values being about 5 to 10 times larger than the cold values. The schematic of a Wien-bridge oscillator using a filament lamp is shown in Figure 1.23. The nonlinearities associated with the lamp resistance provide a challenge in the control of $v_o$.

In oscillators that use op amps the maximum frequency of operation is limited by the frequency response of the op amp. The op amp frequency response is limited by either its gain-bandwidth product or by its slew rate. The gain-bandwidth product is a small-signal limitation, which limits the frequency of oscillation to

$$f_o < \frac{f_T}{|A_{v_0}|} \quad \text{(1.33)}$$

where $f_T$ is the gain-bandwidth product frequency, and $A_{v_0}$ is the midband gain of the amplifier. For an inverting amplifier $A_{v_0} = -R_2/R_1$, and for a noninverting amplifier $A_{v_0} = 1 + R_2/R_1$.

The slew rate is a large-signal limitation, which limits the frequency of oscillation to
Figure 1.22  ADS simulation of the oscillator in Example 1.7.
where SR is the op amp slew rate, and $|v_o|$ is the magnitude of the output voltage.

In the absence of an amplitude-limiting circuit, $|v_o|$ is limited by the op amp saturation. A limiting circuit keeps the operation of the op amp in its linear region (i.e., away from saturation). For a designed value of $|v_o|$, the smaller of the two $f_o$ values in (1.33) or (1.34) limits the maximum oscillation frequency. It also follows that larger values of $f_o$ are obtained, if the amplitude of the oscillation is limited using an amplitude limiting circuit.

### 1.7 The Phase-Shift Oscillator

Phase-shift oscillators usually use RC networks in the feedback path. A phase-shift oscillator is shown in Figure 1.24(a). The op amp is used in an inverting configuration with a gain of $-R_2/R$. Thus, the signal experiences a phase shift of $-180^\circ$ through the amplifier, and the phase shift from each RC section is $60^\circ$ at the frequency of oscillation, for a total phase shift in the feedback path of $180^\circ$. The total phase shift around the closed loop is $0^\circ$ (i.e., $-180^\circ + 180^\circ$). If the gain condition is satisfied, the circuit will oscillate at the frequency where the total phase shift is zero.

The phase-shift network is composed of three RC sections. The following analysis provides insight into the behavior of the phase-shift network. The transfer function of the single RC section shown in Figure 1.25 is

$$
\frac{v_2}{v_1} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\frac{\omega}{\omega_L}}{1 + j\frac{\omega}{\omega_L}}
$$

where

$$\omega_L = \frac{1}{RC}$$
1.7 The Phase-Shift Oscillator

Figure 1.24 (a) A phase-shift oscillator and (b) with RC interchanged.

Figure 1.25 A single RC section.

Hence, the phase shift is

\[ \theta(\omega) = 90^\circ - \tan^{-1} \frac{\omega}{\omega_L} \]  
(1.35)

Equation (1.35) shows that the largest phase shift that can be obtained from a single RC section is 90°. Two RC sections will have a phase shift smaller than 180°. Therefore, a minimum of three RC sections is needed to obtain a phase shift of 180° at the frequency of oscillation.

The phase-shift network in Figure 1.24(a) is described by the feedback factor, namely,
\[
\beta(j\omega) = \frac{v_f}{v_o} = \left( \frac{j \frac{\omega}{\omega_L}}{1 + j \frac{\omega}{\omega_L}} \right)^3 = \frac{-\left( \frac{\omega}{\omega_L} \right)^3}{\omega \left[ 3 - \left( \frac{\omega}{\omega_L} \right)^2 \right] - j \left[ 1 - 3 \left( \frac{\omega}{\omega_L} \right)^2 \right]} \tag{1.36}
\]

The imaginary part of \( \beta(j\omega) \) will vanish at the frequency

\[
\omega_0 = \frac{1}{\sqrt{3}RC}
\]

which is the frequency of oscillation. At the frequency of oscillation it follows from (1.35) that each RC section produces a phase shift of 60°.

From (1.36) the real part of \( \beta(j\omega) \) at \( \omega_0 \) is \( \beta_r = -1/8 \). Therefore, from (1.6), the gain condition is

\[
A_{vo} = \frac{1}{\beta_r} = -8
\]

Hence, it follows that

\[
\frac{R_2}{R} = 8
\]

The phase-shift oscillator in Figure 1.24(a), implemented with \( R \) and \( C \) interchanged in each RC section, is shown in Figure 1.24(b). The resistor \( R \) should be large so that it does not load the third RC section. Of course, a buffer amplifier (i.e., a unity gain amplifier) can be used between the third section and the resistor \( R \) to avoid the loading. In the configuration shown in Figure 1.24(b) the loop gain is

\[
\beta(j\omega)A_{vo} = \frac{A_{vo}}{\left( 1 + j \frac{\omega}{\omega_L} \right)^3}
\]

Hence, it follows that the frequency of oscillation is given by

\[
\omega_0 = \frac{\sqrt{3}}{RC}
\]

and the gain condition is satisfied with \( R_2/R' = 8 \).

Another phase-shift oscillator is shown in Figure 1.26. In this oscillator the RC sections are connected without isolation and, therefore, there is loading. In the last stage the resistors \( R \) and \( R_1 \) appear in parallel. The loading of \( R_1 \) can be neglected if \( R_1 \parallel R = R \), or in some cases by removing \( R \) in the third stage and letting \( R_1 = R \).
The analysis of the phase-shift network in Figure 1.26 with $R_1 \parallel R = R$ gives

$$\beta(j\omega) = \frac{v_f}{v_o} = \frac{1}{\left( \frac{1}{j\omega RC} \right)^3 + 5 \left( \frac{1}{j\omega RC} \right)^2 + 6 \left( \frac{1}{j\omega RC} \right) + 1} \quad (1.37)$$

The imaginary part of $\beta(j\omega)$ comes from the first and third term in the denominator, namely the odd-power terms. The imaginary part of $\beta(j\omega)$ will vanish when

$$\left( \frac{1}{j\omega RC} \right)^3 + 6 \left( \frac{1}{j\omega RC} \right) = 0$$

or at

$$\omega = \omega_o = \frac{1}{\sqrt{6}RC} \quad (1.38)$$

At the frequency of oscillation the phase-shift network produces a phase shift of $180^\circ$.

From (1.37) the real part of $\beta(j\omega)$ at $\omega_o$ is

$$\beta_r(\omega_o) = \frac{1}{5 \left( j\omega_o RC \right)^2 + 1} \quad (1.39)$$

Substituting (1.38) into (1.39) gives

$$\beta_r(\omega_o) = \frac{1}{5(-6) + 1} = -\frac{1}{29}$$

and from (1.6), the gain condition is

$$A_{vo} = \frac{1}{\beta_r(\omega_o)} = -29 \quad (1.40)$$
In Figure 1.26 the resistors $R_1$ and $R_2$ provide an inverting gain of

$$A_{vo} = \frac{v_o}{v_f} = -\frac{R_2}{R_1}$$

Hence, the start of oscillator condition is satisfied if $R_1$ and $R_2$ are selected to provide $|A_{vo}| > 29$.

To summarize, the phase-shift oscillator in Figure 1.26 will oscillate at the frequency $\omega_o$ given by (1.38) if the gain is $|A_{vo}| > 29$. The loading of the op amp is minimized by making $R_1 > 10R$.

The phase-shift oscillator in Figure 1.26 could have been implemented by interchanging $R$ and $C$ in the phase-shift sections. It follows that for such an oscillator the frequency of oscillation is given by

$$\omega_o = \frac{\sqrt{6}}{RC}$$

and the gain condition is $|A_{vo}| > 29$.

Example 1.8
Design the RC oscillator shown in Figure 1.26 to oscillate at 1 kHz.

Solution
Selecting the capacitor values to be 0.1 $\mu$F, then from (1.38)

$$R = \frac{1}{\omega_o \sqrt{6C}} = \frac{1}{(2\pi \times 10^3) \sqrt{6}(0.1 \times 10^{-6})} = 650\Omega$$

The resistors $R_1$ and $R_2$ must provide the gain $|A_{vo}| > 29$ in order to prevent loading $R_1 || R = R$. Letting $R_1 = 15$ k$\Omega$, then $R_2 = 29R_1 = 435$ k$\Omega$. A 495-k$\Omega$ resistor was used to implement $R_2$. This will allow for some extra gain to satisfy the start of oscillation condition (i.e., $|A_{vo}| > 29$).

This oscillator is easy to construct. However, some adjustments in the RC network (due to component variations) are necessary in order to obtain the desired frequency of oscillation. The use of precision resistors and capacitors is recommended. The simulation of this oscillator is shown in Figure 1.27. The fundamental frequency of oscillation is 1.006 kHz. Observe that the output voltage reaches saturation (i.e., $|v_o| = 11$V), since the circuit uses the nonlinearities of the operational amplifier at saturation to determine the amplitude of oscillation. The clipping in the output waveform produces harmonic distortion.

The harmonic distortion can be significantly reduced with an amplitude-limiting circuit. One such oscillator circuit is shown in Figure 1.28. The amplitude-limiting circuit is designed using (1.30) and (1.32), with $v_1$ set equal to zero (i.e., $v_y = 0.7$V and $v_x = 0.7$V).
Example 1.9

Design the phase-shift oscillator in Figure 1.28 to oscillate at 1 kHz. The amplitude of the output voltage is to be limited to \( \pm 5\text{V} \).

Solution

The design of the amplitude-limiting circuit is similar to the one in Example 1.7. From (1.30), with \( v_1 = 0 \) and \( v_o(\text{max}) = 5\text{V} \), we obtain

\[
0.7 = \frac{5R_6}{R_5 + R_6} = \frac{12R_5}{R_5 + R_6}
\]

which can be satisfied with \( R_5 = 2\ \text{k} \Omega \) and \( R_6 = 5.9\ \text{k} \Omega \). From (1.32), for symmetry, we obtain \( R_3 = R_6 = 5.9\ \text{k} \Omega \) and \( R_4 = R_5 = 2\ \text{k} \Omega \).

The simulation is shown in Figure 1.29. The fundamental frequency of oscillation is at 1 kHz, and the total harmonic content of this oscillator is certainly less than that in Figure 1.27.
Figure 1.29  Phase-shift oscillator with an amplitude-limiting circuit.
An interesting phase-shift oscillator is shown in Figure 1.30. It is composed of three integrators with transfer function of

$$A_v(j\omega) = \frac{-1}{\frac{R_1}{j\omega + \frac{1}{R_2C}}}$$

In this oscillator the gain and phase shift is distributed throughout the loop. The loop gain is

$$\beta A_v = \frac{-\left(\frac{1}{R_1C}\right)}{\frac{1}{j\omega + \frac{1}{R_2C}}} = \frac{-\left(\frac{R_2}{R_1}\right)^3}{[1 - 3(\omega R_2C)^2] + j\omega R_2C[3 - (\omega R_2C)^2]}$$

From (1.41) the imaginary part vanishes at the frequency

$$\omega_0 = \frac{\sqrt{3}}{R_2C}$$

and the gain condition follows from setting the loop gain at $\omega = \omega_0$ equal to unity, namely,

$$\beta A_v = \frac{1}{8} \left(\frac{R_2}{R_1}\right)^3 = 1$$

or $R_2 = 2R_1$.

In this oscillator the phase shift from each integrator is $120^\circ$ at $\omega_0$. Since at $\omega_0$ the gain of each stage is $1 [120^\circ]$, the three output voltages are equal in magnitude but at $120^\circ$ from each other.

A quadrature oscillator produces two signals having a phase shift of $90^\circ$. Some phase-shift oscillators can produce signals in quadrature. For example, an additional

![Figure 1.30](image-url)
RC section could have been added to the phase-shift oscillators in Figure 1.24(b), so that each section produces a phase shift of $-45^\circ$, or $-90^\circ$ every two sections. The resulting quadrature oscillator that produces a sine and a cosine signal is shown in Figure 1.31. The amplifier gain is

$$A_{vo} = \frac{v_o}{v_f} = -\frac{R_2}{R'}$$

and the RC phase-shift network transfer function is

$$\beta(j\omega) = \frac{v_f}{v_o} = \frac{1}{\left(1 + j\frac{\omega}{\omega_L}\right)^4}$$  \hspace{1cm} (1.42)

Hence, the loop gain is given by

$$\beta(j\omega)A_{vo} = \frac{-\frac{R_2}{R'}}{\left(1 + j\frac{\omega}{\omega_L}\right)^4}$$  \hspace{1cm} (1.43)

It is observed that at

$$\omega_o = \omega_L = \frac{1}{RC}$$

each RC section produces a phase shift of $-45^\circ$, for a total of $-180^\circ$. Hence, the loop-gain phase shift is $-360^\circ$. At $\omega_o$, (1.42) gives

$$\beta(j\omega) = \frac{1}{(1 + j)^4} = \frac{1}{4} e^{-j\pi}$$

**Figure 1.31** A quadrature oscillator.
Therefore, from (1.43) the required gain is

\[ A_{v0} = -4 \Rightarrow \frac{R_2}{R} = 4 \]

A gain greater than 4 is required to start the oscillation (say, \( A_{v0} = 4.2 \)).

Another quadrature oscillator is shown in Figure 1.32. The second op amp is connected as an inverting integrator; that is,

\[ A_{v2} = \frac{v_{o2}}{v_{o1}} = -\frac{1}{j\omega R_2 C_2} \quad (1.44) \]

This inverter produces a phase shift of \(-270^\circ\) (or \(90^\circ\)).

The first op amp is connected as a noninverting integrator; that is,

\[ A_{v1}(j\omega) = \frac{v_{o1}}{v_f} = 1 + \frac{1}{j\omega R_1 C_1} = \frac{1 + j\omega R_1 C_1}{j\omega R_1 C_1} \quad (1.45) \]

and the feedback network transfer function is

\[ \beta(j\omega) = \frac{v_f}{v_{o2}} = \frac{1}{1 + j\omega R_3 C_3} \quad (1.46) \]

Therefore, from (1.45) and (1.46), with \( R_1 C_1 = R_3 C_3 \), we obtain

\[ \frac{v_{o1}}{v_{o2}} = \frac{1}{j\omega R_1 C_1} \quad (1.47) \]

which shows that the phase shift from \( v_{o2} \) to \( v_{o1} \) is \(-90^\circ\). From (1.44) and (1.47) the loop phase shift is \(0^\circ\) and oscillations occur with \( v_{o2} \) being a sine signal and \( v_{o1} \) a cosine signal.

Using (1.44), (1.45), and (1.46), the loop-gain condition is

![Figure 1.32 A quadrature oscillator.](image)
\[ \beta A_{v1} A_{v2} = \frac{1 + j\omega R_1 C_1}{j\omega R_1 C_1} \left( \frac{1}{j\omega R_2 C_2} \right) \frac{1}{1 + j\omega R_3 C_3} = 1 \]

If \( R_1 C_1 = R_2 C_2 = R_3 C_3 \), the loop-gain condition reduces to

\[ \beta A_{v1} A_{v2} = \frac{1}{(\omega R_1 C_1)^2} = 1 \]

and the frequency of oscillation is

\[ \omega_0 = \frac{1}{R_1 C_1} \]

In a practical quadrature oscillator circuit, the component variations affect the required \( RC \) equality, and therefore, some tuning mechanism is needed, such as a variable potentiometer to implement the resistors. In addition, an amplitude-limiting circuit might be needed to reduce the distortion.

A phase-shift oscillator using a JFET amplifier is shown in Figure 1.33. The input resistance of the amplifier (i.e., \( R_G \)) is very large and does not load the output of the phase-shift network. However, the output resistance of the amplifier is \( r_d || R_D = R_D \) and a certain amount of loading occurs. Good results are obtained by neglecting the loading and designing the oscillator based on (1.38) and (1.40), where \( A_{v0} \approx -g_m R_D \).

A phase-shift oscillator using a BJT amplifier is shown in Figure 1.34(a). The BJT phase-shift oscillator works well at a frequency below 1 MHz. This is the current-controlled equivalent of the JFET phase-shift oscillator. Since the BJT is a current-controlled device, the oscillation frequency and required gain can be obtained using (1.10). The ac model for this circuit is shown in Figure 1.34(b), where it was assumed that \( R_1 || R_2 \gg h_{ie} \) and \( 1/h_{oe} = \infty \). While an approximate

![Figure 1.33](image)
1.7 The Phase-Shift Oscillator

Figure 1.34 (a) A phase-shift oscillator using a BJT amplifier and (b) the phase-shift network.

A design can be made by neglecting the effects of $R_C$ and $h_{ie}$ and the use of a CAD program to optimize the oscillator, it is of interest to show the complexities in the analysis due to their presence in the ac model.

In Figure 1.34(b) the resistor $R'$ is selected so that $R = R' + h_{ie}$. The feedback current is the current into the base of the transistor ($i_f = i_b$), and the input current ($h_{fe}i_b$) is the collector current. From Figure 1.34(b) it can be shown that the loop gain is

$$\beta A_v = \frac{i_f}{i_b} = \frac{-h_{fe}}{3 - \frac{1}{\omega^2 R^2 C^2} + \frac{R}{R_c} - \frac{5}{\omega^2 R R_c C^2} - j \frac{4}{\omega R C} - j \frac{6}{\omega R C} + j \frac{1}{\omega^3 R^2 R_c C^3}}$$

(1.48)

Setting the imaginary part of (1.48) equal to zero gives the frequency of oscillation, namely,

$$\omega_o = \frac{1}{RC \sqrt{6 + \frac{4R_c}{R}}}$$

(1.49)

At $\omega_o$, setting the loop gain in (1.48) equal to unity gives

$$\frac{-h_{fe}}{3 - \frac{1}{\omega_o^2 R^2 C^2} + \frac{R}{R_c} - \frac{5}{\omega_o^2 R R_c C^2}} = 1$$

(1.50)

Substituting (1.49) into (1.50) results in the following gain condition:

$$h_{fe} = 23 + 29 \frac{R}{R_c} + 4 \frac{R_c}{R}$$

(1.51)

This equation can be solved for $R/R_c$ as a function of $h_{fe}$ to determine the minimum value of $R/R_c$ for oscillation. Alternatively, this value can be determined by differentiating (1.51) with respect to $R/R_c$ and setting the result equal to zero. That is,
\[ 29 - 4 \left( \frac{R_c}{R} \right)^2 = 0 \Rightarrow \frac{R}{R_c} = \sqrt{\frac{4}{29}} = 0.372 \]

With \( R/R_c = 0.372 \), (1.49) gives \( h_{fe} = 44.5 \). This is the minimum value of \( h_{fe} \) for oscillations. For \( h_{fe} = 44.5 \) the circuit will oscillate at the frequency given by (1.49). In practical circuits, the transistors’ \( h_{fe} \) are much larger than 44.5, and the tolerances in the resistors and the capacitors usually makes them unequal (especially in the capacitors). Hence, a variable resistor can be used for \( R_c \), and its value changed until the loop gain is slightly larger than unity and the circuit oscillates at the desired frequency with low distortion. The frequency of oscillation will still be closely predicted by (1.49). Another practical way of building this oscillator is to use a variable resistor for \( R' \).

The \( RC \) phase-shift network is one specific case of the general phase-shift network shown in Figure 1.35. It can be shown that for this network

\[ \beta(j\omega) = \frac{v_f}{v_o} = \frac{1}{\left( \frac{Z_1}{Z_2} \right)^3 + 5 \left( \frac{Z_1}{Z_2} \right)^2 + 6 \left( \frac{Z_1}{Z_2} \right) + 1} \] (1.52)

Equation (1.52) shows that oscillations can be obtained with other combinations of \( Z_1 \) and \( Z_2 \), provided that \( \beta_i(\omega) = 0 \). For example, \( Z_1 \) can represent a resistor and \( Z_2 \) an inductor. Of course, the frequency of oscillation depends on the impedances used. Observe that \( Z_1 \) and \( Z_2 \) cannot both be reactive, because in such case \( \beta(j\omega) \) will not have an imaginary part.

1.8 Active-Filter Oscillators

Many oscillators that use an active-filter feedback path have been proposed. Basically, in these oscillators the poles of an active filter are adjusted to lie on the \( j\omega \)-axis at the desired frequency of oscillation. To start the oscillation the poles must be slightly on the right-half plane. A summary of these types of oscillators can be found in [2].

In this section three active-filter oscillators are analyzed to illustrate their design procedure. A simple second-order low-pass filter with positive feedback is shown in Figure 1.36. For this oscillator

![Figure 1.35 A general phase-shift network.](image-url)
Figure 1.36 A simple active-filter oscillator.

\[ A_{v1} = \frac{v_o}{v_f} = -\frac{R_2}{R_1} \]

\[ A_{v2} = \frac{v_b}{v_o} = -\frac{1}{j\omega RC} \]

and

\[ A_{v3} = \frac{v_f}{v_b} = -\frac{1}{j\omega RC} \]

Therefore, the loop gain is

\[ \beta A_{v1} A_{v2} A_{v3} = -\left(\frac{R_2}{R_1}\right)\left(\frac{1}{\omega RC}\right)^2 \]

and the condition for oscillation (i.e., loop gain equal to one) is satisfied at

\[ \omega_o = \frac{1}{RC} \sqrt{\frac{R_2}{R_1}} \]

Next we consider the use of bandpass filters in the implementation of oscillators. Figure 1.37 illustrates an oscillator that uses a bandpass filter in the feedback loop. The voltage transfer function for this filter is

Figure 1.37 An oscillator using a bandpass filter.
Theory of Oscillators

\[
A_{v2} = \frac{v_f}{v_o} = \frac{-\left(\frac{1}{R_1C}\right)^s}{s^2 + \frac{2}{R_2C}s + \frac{1}{R_1R_2C^2}} = \frac{-2Q\omega_0s}{s^2 + \left(\frac{\omega_0}{Q}\right)s + \omega_0^2}
\]

where

\[
Q = \frac{1}{2} \sqrt{\frac{R_2}{R_1}}
\]

and

\[
\omega_o = \frac{1}{C\sqrt{R_1R_2}} \quad (1.53)
\]

The loop gain is

\[
\beta A_{v1}A_{v2} = \left(\frac{R_4}{R_3}\right) \frac{2Q\omega_0\omega}{\omega\left(\frac{\omega_0}{Q}\right) + j(\omega^2 - \omega_0^2)}
\]

which shows that the frequency of oscillation is given by (1.53), and the gain condition is

\[
\frac{R_4}{R_3}2Q^2 = 1 \Rightarrow \frac{R_4}{R_3} = 2 \frac{R_1}{R_2}
\]

An oscillator configuration that uses a Twin-T filter in the feedback path is shown in Figure 1.38. The Twin-T filter consists of two Tee-shaped networks.

Figure 1.38  An oscillator using a Twin-T filter.
1.8 Active-Filter Oscillators

connected in parallel. These Twin-T filters are also known as band-reject filters or notch filters. In the Twin-T filter shown in Figure 1.38, the elements connected to ground have values of $nC$ and $R/n$, respectively.

The transfer function of the Twin-T filter in Figure 1.38 is

$$\beta(s) = \frac{v_f}{v_o} = \frac{s^2 + \left(\frac{2}{n} - 1\right)\omega_0^2}{s^2 + \omega_0 \left(n + \frac{2}{n} + 1\right)s + \omega_0^2} \quad (1.54)$$

where

$$\omega_0 = \frac{1}{RC} \quad (1.55)$$

and $n$ is a number such that $n \geq 2$.

Observe that if $n = 2$, (1.54) reduces to

$$\beta(s) = \frac{s^2 + \omega_0^2}{s^2 + 4\omega_0 s + \omega_0^2} \quad (1.56)$$

which is the transfer function of a Twin-T filter commonly used as notch filter. Of course, the transfer function in (1.56) has a zero at $\omega_0$, which is responsible for the “deep” notch of this filter.

The gain of the op amp is $A_{vo} = -R_2/R_1$. Setting the loop gain of the oscillator equal to 1 gives

$$-A_{vo} \frac{s^2 + \left(\frac{2}{n} - 1\right)\omega_0 s + \omega_0^2}{s^2 + \omega_0 \left(n + \frac{2}{n} + 1\right)s + \omega_0^2} = 1$$

or

$$(1 - A_{vo}) s^2 + \left[n + \frac{2}{n} + 1 - A_{vo} \left(\frac{2}{n} - 1\right)\right] \omega_0 s + (1 - A_{vo}) \omega_0^2 = 0$$

Therefore, the required gain is

$$A_{vo} = -\frac{n + \frac{2}{n} + 1}{1 - \frac{2}{n}} \quad (1.57)$$

and the frequency of oscillation is given by (1.55). Equation (1.57) shows that an oscillator designed with $n = 2$ in the Twin-T filter requires an infinite gain. Hence, for oscillator purposes $n \geq 2.5$ is more practical.
Example 1.10

Design the Twin-T filter in Figure 1.38 to oscillate at $\omega_o = 1 \text{ krad/s}$ (or $f_o = 159 \text{ Hz}$) with $n = 4$.

Solution

Letting $R = 10 \Omega$, it follows from (1.55) that $C = 0.1 \mu \text{F}$. From (1.57), the value of $A_{vo}$ with $n = 4$ is $A_{vo} = -11$. To start the oscillation we let $R_2/R_1 > 11$, or $R_1 = 50 \text{ k}\Omega$ and $R_2 = 660 \text{ k}\Omega$. The op amp supply voltages are $\pm 15\text{V}$.

The simulation of the oscillator is shown in Figure 1.39. The resulting oscillation is at 161.9 Hz. The notch associated with the Twin-T filter (high $Q$) makes the resulting oscillation fairly clean of harmonics.

![Figure 1.39: ADS simulation for the Twin-T oscillator in Example 1.10.](image-url)
References
